

Lambda and mu-symmetries

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Summary. Lambda-symmetries of ODEs were discussed by C. Muriel in her talk at SPT2001. Here we provide a geometrical characterization of λ -prolongations, and a generalization of these – and of λ -symmetries – to PDEs and systems thereof.

Introduction

Symmetry analysis is a standard and powerful method in the analysis of differential equations, and in the determination of explicit solutions of nonlinear ones.

It was remarked by Muriel and Romero [10] (see also the work by Pucci and Saccomandi [14]) that for ODEs the notion of symmetry can be somehow relaxed to that of *lambda-symmetry* (see below), still retaining the relevant properties for symmetry reduction and hence for the construction of explicit solutions. Their work was presented at SPT2001 [11], raising substantial interest among participants.

Here I report on some recent work [4, 6, 7] which sheds some light on “lambda-symmetries”, and extends them to PDEs – and systems thereof – as well; as the central objects here are not so much the functions λ , but some associated one-forms μ , these are called “mu-symmetries”.

The work reported here was conducted together with Giampaolo Cicogna and Paola Morando; I would like to thank them, as well as other friends (J.F. Cariñena, G. Marmo, M.A. Rodríguez) with whom I discussed these topics in the near past. It is also a pleasure to thank C. Muriel and G. Saccomandi for privately communicating their work on λ -symmetries and raising my interest in the topic.

1 Standard prolongations

Let us consider equations with p independent variables $(x^1, \dots, x^p) \in B = \mathbf{R}^p$ and q dependent ones, $(u^1, \dots, u^q) \in F = \mathbf{R}^q$. The corresponding phase space will be $M = B \times F$; more precisely, this is a trivial bundle (M, π, B) .

With the notation $\partial_i := \partial/\partial x^i$ and $\partial_a := \partial/\partial u^a$, a Lie-point vector field in M will be written as

$$X = \xi^i(x, u) \partial_i + \varphi^a(x, u) \partial_a . \quad (1)$$

We also write, with J a multiindex of length $|J| = j_1 + \dots + j_q$, $\partial_a^J := \partial/\partial u_J^a$. Then a vector field in the n -th jet bundle $J^n M$ will be written (sum over J being limited to $0 \leq |J| \leq n$) as

$$Y = \xi^i \partial_i + \Psi_J^a \partial_a^J . \quad (2)$$

The jet space $J^n M$ is equipped with a **contact structure**, described by the **contact forms**

$$\vartheta_J^a := du_J^a - u_{J,i}^a dx^i \quad (|J| \leq n-1) . \quad (3)$$

Denote by \mathcal{E} the $C^\infty(J^n M)$ module generated by these ϑ_J^a . Then we say that Y preserves the contact structure if and only if, for all $\vartheta \in \mathcal{E}$,

$$L_Y(\vartheta) \in \mathcal{E} . \quad (4)$$

As well known, this is equivalent to the requirement that the coefficients in (2) satisfy the (standard) **prolongation formula**

$$\Psi_{J,i}^a = D_i \Psi_J^a - u_{J,m}^a (D_i \xi^m) . \quad (5)$$

We note, for later reference, that for scalar ODEs formula (5) is rewritten more simply, with obvious notation, as

$$\Psi_{k+1} = D_x \Psi_k - u_{k+1} (D_x \xi) . \quad (6)$$

We also recall that the vector field Y is the prolongation of X if Y satisfies (4) and coincides with X when restricted to M ; X is a symmetry of a differential equation (or system of differential equations) Δ of order n in M if its n -th prolongation Y is tangent to the solution manifold $S_\Delta \beta J^n M$, see standard references on the subject [2, 5, 8, 9, 13, 16, 18].

Note that condition (4) is also equivalent to conditions involving the commutator of Y with the total derivative operators D_i ; in particular, it is equivalent to either one of

$$[D_i, Y] \lrcorner \vartheta = 0 \quad \forall \vartheta \in \mathcal{E} ; \quad (7')$$

$$[D_i, Y] = h_i^m D_m + V , \quad (7'')$$

with $h_i^m \in C^\infty(J^n M)$ and V a vertical vector field in $J^n M$ seen as a bundle over $J^{n-1} M$.

2 Lambda-prolongations

2.1 The work of Muriel and Romero

In 2001, C. Muriel and J.L. Romero [10], analyzing the case where Δ is a scalar ODE, noticed a rather puzzling fact.

They substitute the standard prolongation formula (6) with a “lambda-prolongation” formula

$$\Psi_{k+1} = (D_x + \lambda) \Psi_k - u_{k+1} (D_x + \lambda) \xi; \quad (8)$$

here λ is a real C^∞ function defined on $J^1 M$ (or on $J^k M$ if one is ready to deal with generalized vector fields). Let us now agree to say that X is a “lambda-symmetry” of Δ if its “lambda-prolongation” Y is tangent to the solution manifold $S_\Delta \beta J^n M$.

Then, it turns out that “lambda-symmetries” are as good as standard symmetries for what concerns symmetry reduction of the differential equation Δ and hence determination of its explicit solutions. As pointed out by Muriel and Romero, it is quite possible to have equations which have no standard symmetries, but possess lambda-symmetries and can therefore be integrated by means of their approach; see their works [10, 12] for examples.

2.2 The work of Pucci and Saccomandi

In 2002, Pucci and Saccomandi [14] devoted further study to lambda-symmetries, and stressed a very interesting geometrical property of lambda-prolongations: that is, lambda-prolonged vector fields in $J^n M$ can be characterized as the *only* vector fields in $J^n M$ which have the same characteristics as some standardly-prolonged vector field.

We stress that Y is the lambda-prolongation of a vector field X in M , then the characteristics of Y will not be the same as those of the standard prolongation $X^{(n)}$ of X , but as those of the standard prolongation $\tilde{X}^{(n)}$ of a different (for λ nontrivial) vector field \tilde{X} in M .

This property can also be understood by recalling (4) and making use of a general property of Lie derivatives: indeed, for α any form on $J^n M$,

$$L_{\lambda Y}(\alpha) = \lambda Y \lrcorner d\alpha + d(\lambda Y \lrcorner \alpha) = \lambda L_Y(\alpha) + d\lambda \wedge (Y \lrcorner \alpha). \quad (9)$$

2.3 The work of Morando

It was noted [6, 10] that lambda-prolongations can be given a characterization similar to the one discussed in remark 1 for standard prolongations; that is, with h_i^m and V as above, (8) is equivalent to either one of $[D_x, Y] \lrcorner \vartheta = \lambda(Y \lrcorner \vartheta)$ for all $\vartheta \in \mathcal{E}$, and $[D_x, Y] = \lambda Y + h_i^m D_m + V$.

This, as remarked by Morando, also allows to provide a characterization of lambda-prolonged vector fields in terms of their action on the contact forms, analogously to (4). In this context, it is natural to focus on the one-form $\mu :=$

λdx ; note this is horizontal for $J^n M$ seen as a bundle over B , and obviously satisfies $D\mu = 0$, with D the total exterior derivative operator. Then, Y is a lambda-prolonged vector field if and only if $L_Y(\vartheta) + (Y \lrcorner \vartheta)\mu \in \mathcal{E}$ for all $\vartheta \in \mathcal{E}$.

3 Mu-prolongations; mu-symmetries for PDEs

The result given above immediately opens the way to extend lambda-symmetries to PDEs [6]. As here the main object will be the one-form μ , we prefer to speak of “mu-prolongations” and “mu-symmetries”. Let

$$\mu := \lambda_i dx^i \quad (10)$$

be a semibasic one-form on $J^n M, \pi_n, B$, satisfying $D\mu = 0$. Then we say that the vector field Y in $J^n M$ μ -preserves the contact structure if and only if, for all $\vartheta \in \mathcal{E}$,

$$L_Y(\vartheta) + (Y \lrcorner \vartheta)\mu \in \mathcal{E}. \quad (11)$$

Note that $D\mu = 0$ means $D_i \lambda_j = D_j \lambda_i$ for all i, j ; hence locally $\mu = D\Phi$ for some smooth real function Φ .

With standard computations [6], one obtains that (11) implies the **scalar μ -prolongation formula**

$$\Psi_{J,i} = (D_i + \lambda_i) \Psi_J - u_{J,m} (D_i + \lambda_i) \xi^m. \quad (12)$$

Let Y as in (2) be the μ -prolongation of the Lie-point vector field X (1), and write the standard prolongation of the latter as $X^{(n)} = \xi^i \partial_i + \Phi_J \partial_u^J$; note that $\Psi_0 = \Phi_0 = \varphi$. We can obviously always write $\Psi_J = \Phi_J + F_J$, and $F_0 = 0$. Then it can be proved [6] that the difference terms F_J satisfy the recursion relation

$$F_{J,i} = (D_i + \lambda_i) F_j + \lambda_i D_J Q \quad (13)$$

where $Q := \varphi - u_i \xi^i$ is the characteristic [5, 13, 16] of the vector field X .

This shows at once that the μ -prolongation of X coincides with its standard prolongation on the X -invariant space I_X ; indeed, $I_X \beta J^n M$ is the subspace identified by $D_J Q = 0$ for all J of length $0 \leq |J| < n$. It follows that the standard PDE symmetry reduction method [5, 13, 16] works equally well when X is a μ -symmetry of Δ as in the case where X is a standard symmetry of Δ ; see our work [6] for examples.

The concept of μ -symmetries is also generalized to an analogue of standard conditional and partial symmetries [3, 1], i.e. partial (conditional) μ -symmetries [4].

4 Mu-symmetries for systems of PDEs

The developments described in the previous section do not include the case of (systems of) PDEs for several dependent variables, i.e. the case with $q > 1$ in

our present notation. This was dealt with in a recent work [6], to which we refer for details.

To deal with this case, it is convenient to see the contact forms ϑ_J^a , see (3), as the components of a vector-valued contact form [17] ϑ_J . We will denote by Θ the module over q -dimensional smooth matrix functions generated by the ϑ_J , i.e. the set of vector-valued forms which can be written as $\eta = (R_J)_b^a \vartheta_J^b$ with $R_J : J^n M \rightarrow Mat(q)$ smooth matrix functions.

Correspondingly, the fundamental form μ will be a horizontal one-form with values in the Lie algebra $gl(q)$ (the algebra of the group $GL(q)$, consisting of non-singular q -dimensional real matrices) [17]. We will thus write

$$\mu = \Lambda_i dx^i \quad (14)$$

where Λ_i are smooth matrix functions satisfying additional compatibility conditions discussed below.

We will say that the vector field Y in $J^n M$ μ -preserves the vector contact structure Θ if, for all $\vartheta \in \Theta$,

$$LY(\vartheta) + (Y \lrcorner (\Lambda_i)_b^a \vartheta^b) dx^i \in \Theta. \quad (15)$$

In terms of the coefficients of Y , see (2), this is equivalent to the requirement that the Ψ_J^a obey the **vector μ -prolongation formula**

$$\Psi_{J,i}^a = (\nabla_i)_b^a \Psi_J^b - u_{J,m}^b [(\nabla_i)_b^a \xi^m], \quad (16)$$

where we have introduced the (matrix) differential operators

$$\nabla_i := I D_i + \Lambda_i. \quad (17)$$

If again we consider a vector field Y as in (2) which is the μ -prolongation of a Lie-point vector field X , and write the standard prolongation of the latter as $X^{(n)} = \xi^i \partial_i + \Phi_J^a \partial_a^J$ (with $\Psi_0^a = \Phi_0^a = \varphi^a$), we can write $\Psi_J^a = \Phi_J^a + F_J^a$, with $F_0^a = 0$. Then the difference terms F_J satisfy the recursion relation

$$F_{J,i}^a = \delta_b^a [D_i(\Gamma^J)_c^b] (D_j Q^c) + (\Lambda_i)_b^a [(\Gamma^J)_c^b (D_j Q^c) + D_j Q^b] \quad (18)$$

where $Q^a := \varphi^a - u_i^a \xi^i$, and Γ^J are certain matrices (see ref. [6] for the explicit expression). This, as for the scalar case, shows that the μ -prolongation of X coincides with its standard prolongation on the X -invariant space I_X ; hence, again, the standard PDE symmetry reduction method works equally well for μ -symmetries (defined in the obvious way) as for standard ones. See ref. [6] for examples.

5 Compatibility condition, and gauge equivalence

As mentioned above the form μ , see (14), is not arbitrary: it must satisfy a compatibility condition (this guarantees the Ψ_J^a defined by (16) are uniquely

determined), expressed by

$$[\nabla_i, \nabla_k] \equiv D_i \Lambda_k - D_k \Lambda_i + [\Lambda_i, \Lambda_k] = 0. \quad (19)$$

It is quite interesting to remark [4] that this is nothing but the coordinate expression for the horizontal Maurer-Cartan equation

$$D\mu + \frac{1}{2} [\mu, \mu] = 0. \quad (20)$$

Based on this condition, and on classical results of differential geometry [15], it follows that locally in any contractible neighbourhood $A \subseteq J^n M$, there exists $\gamma_A : A \rightarrow GL(q)$ such that (locally in A) μ is the Darboux derivative of γ_A .

In other words, any μ -prolonged vector field is *locally* gauge-equivalent to a standard prolonged vector field [4], the gauge group being $GL(q)$.

It should be mentioned that when $J^n M$ is topologically nontrivial, or μ present singular points, one can have nontrivial μ -symmetries; this is shown by means of very concrete examples in our recent work [4].

Note that when we consider symmetries of a given equation Δ , the compatibility condition (20) needs to be satisfied only on $S_\Delta \subseteq J^n M$. When indeed μ is not satisfying everywhere (20), μ -symmetries can happen to be gauge-equivalent to standard *nonlocal symmetries* of exponential form; see again ref. [4] for details.

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